$x' = Ax \quad (1)$

~~first case~~ A is $n \times n$ matrix

① if A has n distinct eigenvalues $r_1 \cdots r_n$

then the general sol of (1) is

$x = c_1 \xi_1 e^{r_1 t} + \cdots + c_n \xi_n e^{r_n t}$

ξ_i is the eigenvector corresponding to r_i

Note that $\{\xi_i\}$ is linear independent since $r_i \neq r_j$

② A has repeated eigenvalues $r_1 \cdots r_s$

with multiplicity $n_1 \cdots n_s$ $\quad n_1 + \cdots n_s = n$

(1) if r_i has n_i linearly independent eigenvectors $\xi_{n_1}^{(i)} \cdots \xi_{n_i}^{(i)}$

Then we have n_i independent sol. $x_{n_1}^{(i)} = \xi_{n_1}^{(i)} e^{r_i t} \cdots x_{n_i}^{(i)} = \xi_{n_i}^{(i)} e^{r_i t}$

~~(2) if~~ In (1) case, A can be diagonalized.

i.e. $\exists T$ s.t. $T^{-1}AT = \begin{pmatrix} r_1 & & \\ & r_{i} & \\ & & r_s \end{pmatrix} = D$

we can solve in another way $\quad x = Ty$

then $y' = T^{-1}ATy \Rightarrow \underline{y = \exp(Dt)\, y_0}$

(2) if ~~r_i has~~ the number of eigenvectors of r_i less than n_i.

~~say n_i~~ For example r_i only has 1 eigenvector $\xi_{n_1}^{(i)}$, we have a sol $x_{n_1}^{(i)} = \xi_{n_1}^{(i)} e^{r_i t}$

then we can generate other $(n_i - 1)$ sol by following ways.

we assume $x_{n_2}^{(i)} = \xi_{n_1}^{(i)} t e^{r_i t} + \xi_{n_0}^{(i)} e^{r_i t}$ $\quad$ Similarly, for $x_{n_k}^{(i)}$

where $\xi_{n_1}^{(i)}, \xi_{n_0}^{(i)}$ satisfy

$$\begin{cases} (A - r_i I)\, \xi_{n_i}^{(i)} = 0 \\ (A - r_i I)\, \xi_{n_0}^{(i)} = \xi_{n_i}^{(i)} \end{cases}$$

we assume $x_{n_k}^{(i)} = \frac{t^k e^{r_i t}}{k!} \xi_{n_k}^{(i)} + \frac{t^{k-1} e^{r_i t}}{(k-1)!} \xi_{k-1}^{(i)} \cdots + \xi_{n_0}^{(i)} e^{r_i t}$

when $\cdots$ satisfy

$$\begin{cases} (A - r_i I)\, \xi_{n_k}^{(i)} = 0 \\ (A - r_i I)\, \xi_{n_{k-1}}^{(i)} = \xi_{n_k}^{(i)} \\ \vdots \\ (A - r_i I)\, \xi_{n_0}^{(i)} = \xi_{n_1}^{(i)} \end{cases}$$

Then,

$$\mathbf{T}\exp(\mathbf{J}t) = \begin{pmatrix} 1 & 2 & 0 \\ 0 & 4 & 0 \\ 2 & -2 & -1 \end{pmatrix} \begin{pmatrix} e^t & 0 & 0 \\ 0 & e^t & te^t \\ 0 & 0 & e^t \end{pmatrix} = \begin{pmatrix} e^t & 2e^t & 2te^t \\ 0 & 4e^t & 4te^t \\ 2e^t & -2e^t & -(2t+1)e^t \end{pmatrix}$$

It is as the fundament matrix $\Psi(t)$ in Problem 18(e). ❑

Note that. In this case we write the n_t s.t.

$x_{n_1}^{(i)} = e^{\lambda_i t} \xi_{n_1}^{(i)}$

$x_{n_2}^{(i)} = e^{\lambda_i t}\left(\xi_{n_1}^{(i)} t + \xi_{n_2}^{(i)}\right)$

$\vdots$

$x_{n_k}^{(i)} = e^{\lambda_i t}\left(\frac{t^k}{k!}\xi_{n_k}^{(i)} + \cdots + \xi \cdots + \xi_{n_2}^{(i)}\right)$

The $\xi_{n_0}^{(i)} \cdots \xi_{n_k}^{(i)} \cdots \xi_{n}^{(i)}$ are same since λ_i has one eigenvalue.

eg. if $n=3$, triple eigenvalue λ. and λ has 2 eigenvectors. ξ_1, ξ_2 we get two ~~eign~~ s.t $x_1 = \xi_1 e^{\lambda t}$, $x_2 = \xi_2 e^{\lambda t}$.

to find a third s.t. we ass $x_3 = \xi_3 t e^{\lambda t} + y e^{\lambda t}$

where $\begin{cases} (A-\lambda I)\xi_3 = 0 & (2) \\ (A-\lambda I) y = \xi_3 & (3) \end{cases}$ ← be carefully: ξ_3 can't be chose arbitary

Since (3) ~~can~~ may not has a sol., we can solve this by note that $(A-\lambda I)^2 y = 0$ and $(A-\lambda I)^2 = 0$ because hence y can be choose arbitary but should be indep of ξ_1 and ξ_2 otherwise $(A-\lambda I) y = 0$.

Another way for repeated eigenvalues which A can't be diagonalized.

if the multiplicity of r_i is n_i.

then we will have n_i L-I. set

$$x_{1}^{(i)} = \left(\xi_{11}^{(i)} + \xi_{12}^{i} t + \dots + \xi_{1n_i}^{i} \frac{t^{n_i}}{n_i!}\right) e^{r_i t}$$

$$x_{2}^{(i)} = \left(\xi_{21}^{(i)} + \xi_{22}^{i} t + \dots + \xi_{2n_i}^{i} \frac{t^{n_i}}{n_i!}\right) e^{r_i t}$$

$\vdots$

$$x_{n_i}^{(i)} = \left(\xi_{n_i 1}^{(i)} + \xi_{n_i 2}^{i} t + \dots + \xi_{n_i n_i}^{(i)} \frac{t^{n_i}}{n_i!}\right) e^{r_i t}$$

where $\xi_{j1}^{(i)}$ $(j=1,\dots n_i)$ is n_i L.I vector satisfy

$(A-r_i I)^{n_i} \xi = 0$

$$\begin{cases} \xi_{j2}^{(i)} = (A-r_i I)\,\xi_{j1}^{(i)} \\ \xi_{jk}^{(i)} = (A-r_i I)\,\xi_{jk-1}^{(i)} \end{cases}$$

Can be simplified

eg, suppose A is 3x3 matrix has a triple eigenvalues λ.

the has 2- eigenvectors L-I corresponding to λ η_1, η_2

we assume $x_1 = (\xi_{11} + \xi_{12} t + \xi_{13} \frac{t^2}{2}) e^{\lambda t}$ $\xi_{11}, \xi_{21}, \xi_{31}$ are L-I set of

$x_2 = \dots$ $(A-\lambda I)^3 \xi = 0$ but since

$x_3 = (\xi_{31} + \dots \xi_{33} \frac{t^2}{2}) e^{\lambda t}$ $(A-\lambda I)^3 = (A-\lambda I)^2 = (0)_{3\times 3}$

Which means, we can choose ~~ξ_{11}, there are~~ two ~~vectors~~ are linear comb. of

$y_1, y_2 \xleftarrow{\text{denotes}} \xi_{11}, \xi_{21}$

then since y_2 is sol of $(A-\lambda I)y=0$

we have $\xi_{12} = (A-\lambda I)\xi_{11} = (A-\lambda I)(c_{11}y_1 + c_{12}y_{2}) = 0$

$\xi_{22} = \cdots = 0$

Similarly $\xi_{13} = \xi_{23} = 0$

for ξ_{31}. $(A-\lambda I)\xi_{31} \neq 0 = \xi_{32}$

but since $\underline{(A-\lambda I)^2 = 0}$ $\xi_{33} = (A-\lambda I)\xi_{32} = (A-\lambda I)^2\xi_{31} = 0$.

Hence

$$\boxed{\begin{aligned} x_1 &= \xi_{11}e^{\lambda t} \\ x_2 &= \xi_{21}e^{\lambda t} \\ x_3 &= \xi_{31}e^{\lambda t} + \xi_{32}te^{\lambda t} \end{aligned}}$$

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proof $\Leftarrow$ ① ~~{ } linearly dependent~~

② te^{λ_2} $t^2e^{\lambda_3}\cdots$

proof $\Leftarrow$ $\{\varphi_1 \cdots \varphi_n\}$ is a fundamental set

Jordan form

eg. A 3×3 matrix, has a triple eigenvalues $\lambda=1$
has two l.I eigenvectors corresponding to $\lambda=1$

Then the Jordan form of A is $\begin{pmatrix}1&&\\&1&1\\&&1\end{pmatrix}=D$

$$\exp(Dt)=I+\sum_{n=1}^{\infty}\frac{(Dt)^n}{n!}$$

$$D^n=\begin{pmatrix}1^n&&\\&1^n&n1^{n-1}\\&&1^n\end{pmatrix}=\begin{pmatrix}1&&\\&1&n\\&&1\end{pmatrix}$$

$$\begin{pmatrix}&0\\&0\\&0\end{pmatrix}\begin{pmatrix}&&\end{pmatrix}=\begin{pmatrix}0\\0\\0\end{pmatrix}$$

$$\exp(Dt)=\begin{pmatrix}e^t&&\\&e^t&te^t\\&&e^t\end{pmatrix}$$

$\Psi(t)=T\exp(Dt)$

$\exp(At)=T\exp(Dt)T^{-1}$

Which means the system $x'=Ax$
has 3 l.I sol. $x_1=\xi^1 e^t$ $x_2=\xi^2 e^t$ $x_3=\xi^3 e^t+\eta t e^t$

eg 2. A is 3×3 matrix has a triple eigenvalues $\lambda=1$.
which has only one eigenvector corresponding to $\lambda=1 \Rightarrow \eta$

Then the Jordan form of A is $\begin{pmatrix}1&1&\\&1&1\\&&1\end{pmatrix}$

We assume
$x_1=(\xi_{11}+\xi_{12}t+\xi_{13}\frac{t^2}{2})e^t$
$x_2=(\xi_{21}+\xi_{22}t+\xi_{23}\frac{t^2}{2})e^t$
$x_3=(\xi_{31}+\xi_{32}t+\xi_{33}\frac{t^2}{2})e^t$

ξ_1, ξ_2, ξ_3 are l.I sols of $(A-\lambda I)^3\xi=0$

Since $(A-\lambda I)^3=(0)_{3\times3}$, we can choose ξ_1 to be η. Then since $x_1=\eta e^t$

for x_2 and x_3 since $\ker(A-\lambda I)^2=2$.

~~$\xi_{23} = (A-\lambda I)^2 \xi_{41}$ $\to$ either $\xi_{23}=0$, or $\xi_{33}=0$~~

~~$\xi_{33} = (A-\lambda I)^2 \xi_{31}$ Since ξ_{21} and ξ_{31} are lin. indep.~~

We can choose $\xi_{21} \in \ker(A-\lambda I)^2 / \ker(A-\lambda I)$, $\xi_{31} \in \ker(A-\lambda I)^3 / \ker(A-\lambda I)^2$.

Then $\xi_{23} = (A-\lambda I)^2 \xi_{21} = 0$ $\xi_{33} \neq 0$

Here $x_1 = \eta e^t$

$x_2 = (\xi_{21} + \xi_{22} t) e^t$

$x_3 = (\xi_{31} + \xi_{32} t + \xi_{33} \frac{t^2}{2}) e^t$.

The Jordan form of A is $\begin{pmatrix} 1 & 1 & \\ & 1 & 1 \\ & & 1 \end{pmatrix} = D$ $P^{-1}AP = D$.

$$D^n = \begin{pmatrix} 1^n & n 1^{n-1} & \frac{n(n-1)}{2} 1^{n-2} \\ & 1^n & n 1^{n-1} \\ & & 1^n \end{pmatrix} \Rightarrow \exp(Dt) = \begin{pmatrix} e^t & te^t & \frac{t^2}{2} e^t \\ & e^t & te^t \\ & & e^t \end{pmatrix}$$

$\varphi(t) = P \exp(Dt) = \begin{pmatrix} \end{pmatrix} \cdot \begin{pmatrix} \end{pmatrix} = \begin{pmatrix} \end{pmatrix} \begin{pmatrix} \end{pmatrix} \begin{pmatrix} \end{pmatrix}$ contain $\frac{t^2}{2} e^t$

Abel's formula for systems of first order

and high order eq.